

Representability of diagonals & examples

Importance of diagonal: diagonal encodes the stackiness!

Thm: 1) X alg space $\Rightarrow X \xrightarrow{\Delta} X \times X$ is repn. by schemes
 2) $\mathcal{X}$ alg stack $\Rightarrow \mathcal{X} \xrightarrow{\Delta} \mathcal{X} \times \mathcal{X}$ is representable.

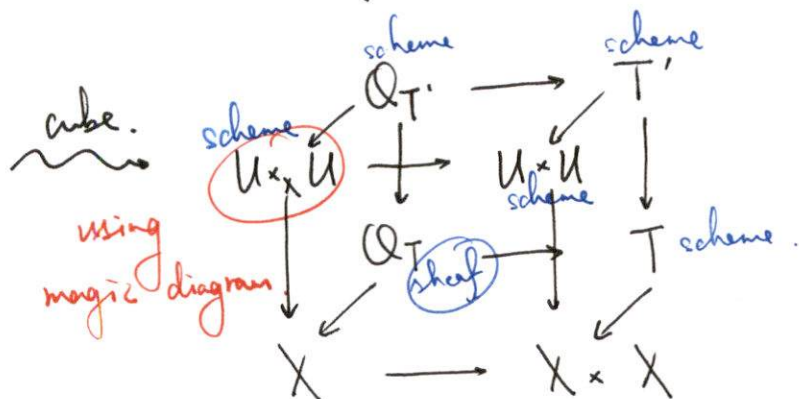
pf): 1) $Q_T \rightarrow T$ scheme
 $\downarrow \quad \square \quad \downarrow$
 $X \xrightarrow{\Delta} X \times X$

want: Q_T is a scheme.

scheme
 $T' \rightarrow T$ *scheme*
 $\downarrow \quad \square \quad \downarrow$

scheme
 $U \xrightarrow{\tilde{\sigma}_T} X$ étale presentation $\Rightarrow U \times U \rightarrow X \times X$

étale surj. upn. by schemes



all are Cartesian!

$U \times_X U \rightarrow U \times U$ separated & loc. g-finite.

$\Rightarrow$ so is $Q_{T'} \rightarrow T'$

eff. descent $\Rightarrow Q_T$ is a scheme.

2) X alg stack. $U \xrightarrow{\text{scheme}} X$ sm. presentation.

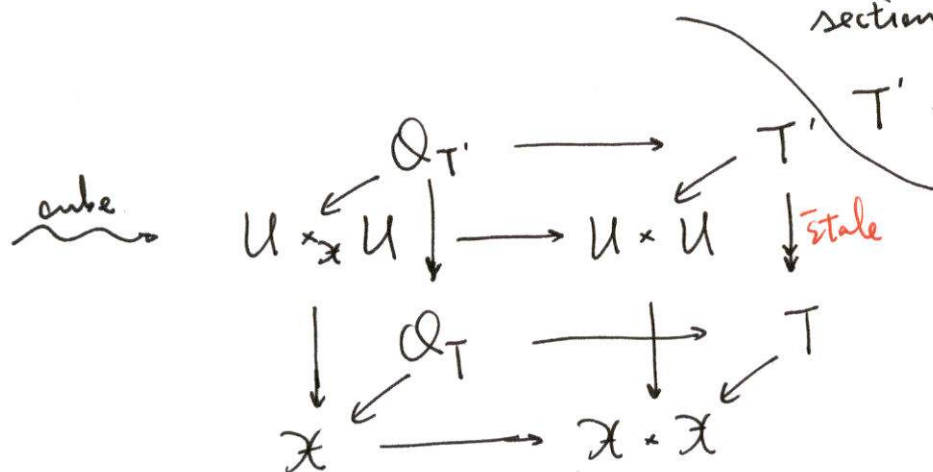
$U \times_X U$ alg. space.

$T \xrightarrow[\text{pre.}]{\text{st.}} T_1 \xrightarrow{\text{alg. space.}} T$ scheme

$$\begin{array}{ccc} \downarrow & \square & \downarrow \\ U \times U & \longrightarrow & X \times X \end{array}$$

$T_1 \rightarrow T$ is a sm. surj.

morphism of schemes w/ a section after an étale cover



All are Centesim

scept for left & right

$\Rightarrow Q_{T'} \rightarrow Q_T$ surj. ~~stale~~ ^{stale} repn. by schemes, since $T' \rightarrow T$ is

$V \rightarrow Q_{T'}$ stale presentation $\rightsquigarrow V \xrightarrow{\text{st}} Q_T$ stale presentat

□

Remark: (magic diagram) $X_1, X_2 \rightarrow Y \rightarrow Z$

$$\begin{array}{ccc} X_1 \times_Y X_2 & \longrightarrow & X_1 \times_Z X_2 \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\Delta} & Y \times_Z Y \end{array}$$

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Cor: 1) any morphism from a scheme to an alg. space is repn. by schem
 2) alg. stack is representable

pf: $T_1 \times_{\mathcal{X}} T_2 \rightarrow T_2$ ^{scheme}
 $\downarrow \quad \downarrow$
 T_1 ^{scheme} $\xrightarrow{\text{presentation}}$ $\mathcal{X}$

$T_1 \times_{\mathcal{X}} T_2 \rightarrow \mathcal{X} T_1 \times T_2$ ^{scheme}
 $\downarrow \quad \downarrow$
 $\mathcal{X} \xrightarrow{\Delta} \mathcal{X} \times \mathcal{X}$ ^{repr (by schemes)}

Exercises: i) $\mathcal{X} \rightarrow \mathcal{Y}$ morphism of alg. stacks $\Rightarrow \mathcal{X} \rightarrow \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ representable & loc. of finite type by scheme
 ii) $\mathcal{X} \rightarrow \mathcal{Y}$ representable $\Rightarrow$ by scheme
 iii) map

Q: properties of diagonals?

Key: important feature: diagonal encodes the stabilizer groups!

Defn: $\mathcal{X}$: alg. stack. $x \in \mathcal{X}(k)$ k -valued pt. The stabilizer
 of x is $G_x \rightarrow \text{Spec } k$ as an alg. grp
 $\downarrow \quad \downarrow (x, x)$ a grp alg space
 $\mathcal{X} \xrightarrow{\Delta} \mathcal{X} \times \mathcal{X}$ (as $\mathcal{X} \rightrightarrows \mathcal{X} \times \mathcal{X}$ repn.)

Rem: grp alg space = an alg. space w/ a grp structure (mult. inverse. ^{identity} identity)

Exercise: (motivation) $G: \text{group scheme}/k \rightarrow X: \text{scheme}/k, x \in X(k)$

$\Rightarrow$ the stabilizer $G_x = \text{usual stabilizer group scheme.}$

pf) Recall:

$$\begin{array}{ccc}
 G \times X & \xrightarrow{G_x} & X \times X \\
 \downarrow & \square & \downarrow \\
 [X/G] & \xrightarrow{\Delta} & [X/G] \times [X/G]
 \end{array}$$

Red arrows and labels: G_x (top left), (x, x) Spuck (top right), $(\bar{x}, \bar{x})$ (bottom right).

Ex: (3.2.8) $\mathcal{X}: \text{DM stack}$ (g-separated)

a) $\forall x \in \mathcal{X}(k)$. G_x is a separated étale (finite) group scheme/k.

b) if $k = \bar{k}$. then G_x is discrete & reduced. (finite)

c) $\Delta: \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ is unramified.

Now: varying $x \in \mathcal{X}$. $\rightsquigarrow G_x$ varies in a family.

Defn: (Inertia stack)

$$\begin{array}{ccc}
 I_{\mathcal{X}} & \rightarrow & \mathcal{X} \\
 \downarrow & \square & \downarrow \Delta \\
 \mathcal{X} & \xrightarrow{\Delta} & \mathcal{X} \times \mathcal{X}
 \end{array}$$

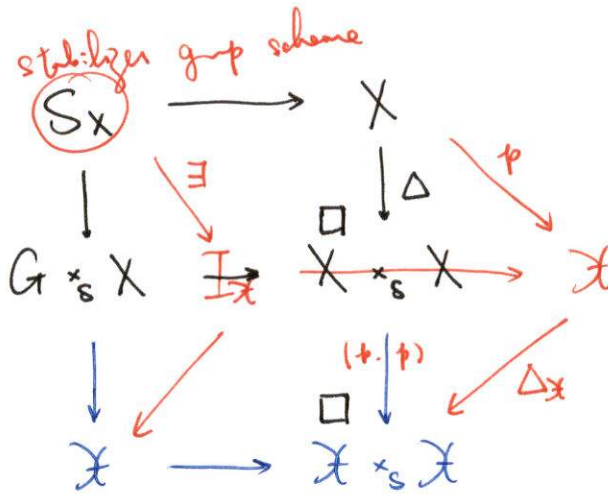
The fibre of $I_{\mathcal{X}} \rightarrow \mathcal{X}$ over a field-valued pt $x \in \mathcal{X}(k)$ is exactly G_x .

Exercise / Example: $G \rightarrow S$ grp scheme $\hookrightarrow X \rightarrow S$ scheme.

$X = [X/G]$. quot. stack. Then

$$\begin{array}{ccc} S_X & \longrightarrow & X \\ \downarrow & & \downarrow \textcolor{red}{p} \\ I_X & \longrightarrow & \mathcal{X} \end{array} \quad \square$$

where



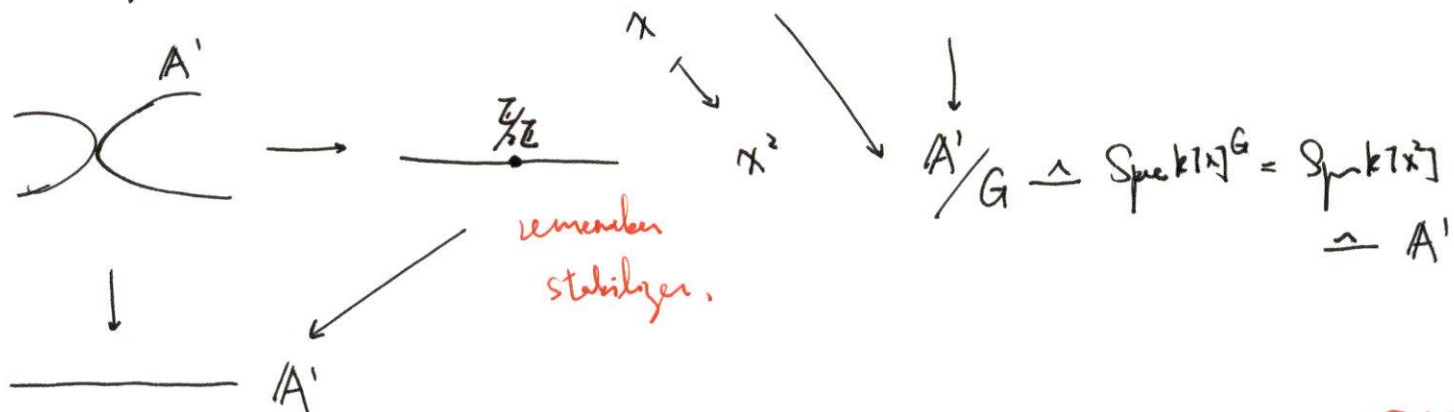
Ex 13.7.13: G : finite abelian grp. $\curvearrowright$ U scheme

$$\Rightarrow I_{[U/G]} = \bigsqcup_{g \in G} [U^g/G] \quad . \quad U^g := \{x \in U : gx = x\}$$

types of spaces	properties of diagonal	properties of stabilizer
separated schemes	closed immersion	trivial.
schemes	immersion	trivial
alg. spaces	monomorphism	trivial
DM stacks	unramified	discrete & reduced groups
alg. stacks	arbitrary	arbitrary.

First examples:

1) $G = \mathbb{Z}/2\mathbb{Z} \curvearrowright A'$



2) $G = G_m \curvearrowright A'$



3) $G_m \curvearrowright A^2$



$[A^2/G_m] = \mathbb{P}^1 \sqcup \{pt\}$

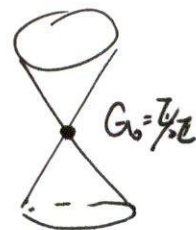
$A^2 \rightarrow [A^2/G_m]$

$A^2/G_m = \text{Spec } k[x,y]^G = \{pt\}$

only closed pt
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4) $G = \mathbb{Z}/2\mathbb{Z} \curvearrowright \mathbb{A}^2$ $(-1) \cdot (x, y) = (-x, -y)$

$\mathbb{A}^2 \rightarrow [\mathbb{A}^2/G]$ *sm. DM stack*



$\downarrow$
 $\mathbb{A}^2/G = \text{Spec } k[x^2, xy, y^2] = \text{affine cone} / \text{conic } \mathbb{CP}^2$



5) (Hironaka's example) $\exists$ *but not vj.* sm. proper 3-fold X w/ a free $\mathbb{Z}/2\mathbb{Z}$ -action

s.t. $\exists$ orbit not contained in any affine open.

$\leadsto X \rightarrow X/(\mathbb{Z}/2\mathbb{Z})$ is an alg. space but not a scheme.

A quick introduction on Equivariance relation & groupoid geom.

An étale/sm. groupoid of schemes is a pair of étale/sm.

maps $R \xrightleftharpoons[r]{s} U$ of schemes, *source/target*, & a

composition morphism $R \times_{r, U, s} R \xrightarrow{c} R$ satisfying.

① (associativity) $R \times_{r, U, s} R \times_{r, U, s} R \xrightarrow{c \circ \text{Id}} R \times_{r, U, s} R$

$\downarrow \text{Id} \times c \quad \quad \quad \downarrow c$
 $R \times_{r, U, s} R \xrightarrow{c} R$

② (Identity) $\exists e: U \rightarrow R$ s.t.

$$\begin{array}{ccccc} & & U & & \\ & \swarrow \text{id} & \downarrow e & \searrow \text{id} & \\ U & \xleftarrow{s} & R & \xrightarrow{t} & U \end{array}$$

$$\begin{array}{ccccc} R & \xrightarrow{(e \circ s, \text{id})} & R \times_{t, U, s} R & \xleftarrow{(\text{id}, e \circ t)} & R \\ & \searrow \text{id} & \downarrow c & \swarrow \text{id} & \\ & & R & & \end{array}$$

③ (Inverse) $\exists i: R \rightarrow R$ s.t.

$$\begin{array}{ccccc} R & \xrightarrow{i} & R & \xleftarrow{i} & R \\ & \searrow s & \downarrow \tau & \swarrow s & \\ & & U & & \end{array}$$

$$\begin{array}{ccc} R & \xrightarrow{s/\tau} & U \\ \downarrow (\text{id}, i) / (i, \text{id}) & & \downarrow e \\ R \times_{t, U, s} R & \xrightarrow{\quad} & R \end{array}$$

If moreover $(s, t): R \rightarrow U \times U$ is a monomorphism, then we call $s, t: R \rightrightarrows U$ an *étale/sm equivalence relation*.

Example: $G: \text{sm. alg. grp}/k \xrightarrow{s} U := k\text{-scheme}$

$$R := G \times U \xrightleftharpoons[t = p_2]{s = s} U \quad \text{sm. (étale if } G \text{ is finite)}$$

$$R \times_{t, U, s} R = \{(g_1, u_1), (g_2, g_1, u_1)\} \xrightarrow{c} R = G \times U$$

$$e: U \rightarrow R = G \times U$$

$$u \mapsto (1, u)$$

$$i: R \rightarrow R$$

$$(g, u) \mapsto (g^{-1}, gu)$$

Equivalence relation $\Leftrightarrow G \curvearrowright U$ freely, i.e. $G \times U \rightarrow U \times U$ mono.

Idea: think of R as a "scheme of relations"

$$r \in R \rightsquigarrow "s(r) \xrightarrow{r} t(r)"$$

- composition : $(u \xrightarrow{r} v) \circ (v \xrightarrow{r'} w) = (u \xrightarrow{r' \circ r} w)$
- identity : $(u \xrightarrow{\text{id}} u)$
- inverse : $(u \xrightarrow{r} v) \rightsquigarrow (v \xrightarrow{r^{-1}} u)$

$R \rightrightarrows U$ eq. rel. $(\Rightarrow) \exists$ at most one rel. b/w. any
 $\geq$ (ordered) pt of U .

Reason: $\rightsquigarrow$ groupoid of germs.

$$R \rightrightarrows U \text{ sm. groupoid.} \rightsquigarrow [U/R]^{\text{pre}} \begin{cases} \text{obj} = "T - U" \\ \text{morphisms: } \text{Mor}\left(\begin{smallmatrix} S \\ \downarrow a \\ U \end{smallmatrix}, \begin{smallmatrix} T \\ \downarrow b \\ U \end{smallmatrix}\right) = \end{cases}$$

$$\left\{ (S \xrightarrow{f} T, r \in R(S)) \mid a = s(r), \text{ } \frac{b}{t(m)} = b \circ f \right\}$$

$$[U/R] := ([U/R]^{\text{pre}})^{\text{st.}}$$

Thm: $R \rightrightarrows U$ étale (sm.) groupoid of schemes. Then $[U/R]$ is

a DM (alg.) stack, and $U \rightarrow [U/R]$ is an étale (sm.) presentation

Further examples:

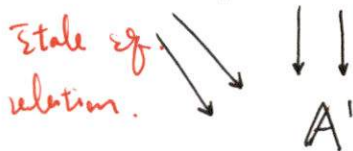
6) (big-syed cover) 2 descriptions

② $\mathbb{Z}/2\mathbb{Z} \curvearrowright A' \cup_{A'/\sim} A' =: U$  non-sep. affine line

$$X := U / (\mathbb{Z}/2\mathbb{Z})$$

⑥ $\mathbb{Z}/2\mathbb{Z} \curvearrowright A'$ $(1,1)x = x$

$R \curvearrowright R \subset \mathbb{Z}/2\mathbb{Z} \times A'$ $R = (\mathbb{Z}/2\mathbb{Z} \times A') \setminus \{(-1,0)\}$

stale eq. relation. 

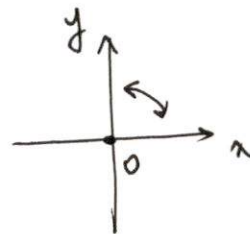
$$X := A'/R \text{ alg. space.}$$

NB: X is not a scheme. $\{(-1,z) | z \neq 0\} \cup \{(1,0)\} \xrightarrow{\varphi} A'$

$\varphi: \begin{array}{c} | \\ \bullet \\ | \\ \text{Gen } \cup \{0\} \end{array} \rightarrow \begin{array}{c} | \\ | \\ | \\ A' \end{array}$ not loc. closed immersion.

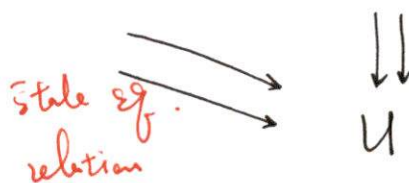
$\begin{array}{ccc} R & \rightarrow & A' \times A' \quad (z, -z) \\ \downarrow & & \downarrow \\ X & \xrightarrow{\Delta} & X \times X \end{array}$ NOT loc. closed.

7) $\mathbb{Z}/2\mathbb{Z} \curvearrowright U = \text{Spec } k[x,y]/(x,y)^2$



$(1,1) \cdot (x,y) \stackrel{\sim}{\mapsto} (y,x)$

$R = \mathbb{Z}/2\mathbb{Z} \times U \setminus \{(-1,0)\} \subset \mathbb{Z}/2\mathbb{Z} \times U$

stale eq. relation 

$X = U/R$

$X = \text{---} \bullet \text{---} A'_k$ not sm.

scheme? 

8) $G_m \curvearrowright A^2$
 $t \cdot (x, y) = (tx, t^2y)$

$\rightsquigarrow \mathcal{X} = [A^2/G_m]$
 $\downarrow$
 $A^2/G_m = A^1$


 non-sep. affine line $\mathbb{A}^1_{\text{sp}}$
 (sm). alg stack

9) char $k=0$ $\mathbb{Z} = \bigsqcup_{n \in \mathbb{Z}} \text{Spec } k$ a gpt scheme / k

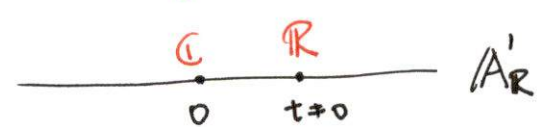
$\mathbb{Z} \curvearrowright A'_k$ via $n \cdot x := x + n$

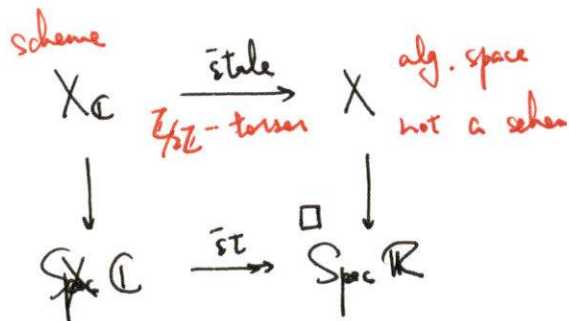
$\rightsquigarrow A'_k/\mathbb{Z}$ is an alg. space but not a scheme. NOT quasi-sep

Since $\mathbb{Z} \times A'_k \not\rightarrow A'_k \times A'_k$ not quasi-compact.

10) $\mathbb{Z}/2\mathbb{Z} \curvearrowright A'_\mathbb{C}$ via conjugation $\mathbb{C} \rightarrow \mathbb{C}$ defined / $\mathbb{R}$
 $z \mapsto \bar{z}$

$R = \mathbb{Z}/2\mathbb{Z} \times A'_\mathbb{C} \setminus \{1, 0\} \xrightarrow[\text{relation}]{\text{stable eq.}} A'_\mathbb{C} \rightarrow X := A'_\mathbb{C}/R \quad (/ \mathbb{R})$

$X:$  residue field



Claim: $X_\mathbb{C} = \text{non-sep. affine line} / \mathbb{C}$

 swapping origin & conjugation on $A'_\mathbb{C} \setminus \{0\}$